***Counting + Principle of Inclusion–Exclusion***

**Problem 1**

**Part A:** How many permutations of the letters ABCDEFGH contain

a) the string ED? b) the string CDE? c) the strings BA and FGH? d) the strings AB, DE, and GH? e) the strings CAB and BED? f ) the strings BCA and ABF?

Solution:

1. The set of permutations of the letters include of which there are
2. The set of permutations of the letters ABCDEF GH that contain the string CDE is the same as the set of permutations of the 6-element set . The latter set has elements
3. Here we are interested in the set of permutations of the set , of which there are
4. Here, are interested in the set of permutations of the set of which there are elements.
5. B cannot occur twice therefore, there are zero permutations.
6. B cannot occur twice therefore, there are zero permutations

**Part B:** The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel? b) exactly two vowels? c) at least one vowel? d) at least two vowels?

Solution:

1. Exactly one vowel

# of strings = )

1. Exactly two vowels

# of strings = )

1. At least one vowel

# of strings =

1. At least two vowels

# of strings =

**Problem 2**

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Solution: Such a committee can be formed by 3 men and 3 women. Therefore, we can find the members as,

**Problem 3**

You want to send postcards to 12 friends. In the shop there are only 3 kinds of postcards. In how many ways can you send the postcards, if

1. there is a large number of each kind of postcard, and you want to send one card to each friend;

Solution: Suppose, we have three types of postcards as A, B, and C. As we have 3 choices for each friend, therefore, ways to send the postcard.

1. there is a large number of each kind of postcard, and you are willing to send one or more postcards to each friend (but no one should get two identical cards);

Solution: Suppose, we have three types of postcards as A, B, and C. We can send postcards to the first friend in three ways. One, either we can send him three cards as either A or B or C. Second, either send A and B, or A and C, or B and C. Three, we can send all three cards, A and B and C. Therefore there are total 7 choices for the first friend. Same choices are available for all friends. Total ways will be

1. the shop has only 4 of each kind of postcard, and you want to send one card to each friend?

Solution: Suppose, we have three types of postcards as A, B, and C. We have total 12 cards as each card type has four cards.

For first type of cards, we can choose 4 friends out of 12: ways.

For second type of cards, we can choose 4 friends out of 8: ways.

For third type of cards, we can choose 4 friends out of 4: ways.

Final answer:

**Problem 4**

There is a class of 40 girls. There are 18 girls who like to play chess, and 23 who like to play soccer. Several of them like biking. The number of those who like to play both chess and soccer is 9. There are 7 girls who like chess and biking, and 12 who like soccer and biking. There are 4 girls who like all three activities.

In addition we know that everybody likes at least one of these activities. How many girls like biking?

Solution:

**Problem 5**

1. How many positive integers not bigger than 20 are divisible by either 2 or 3?

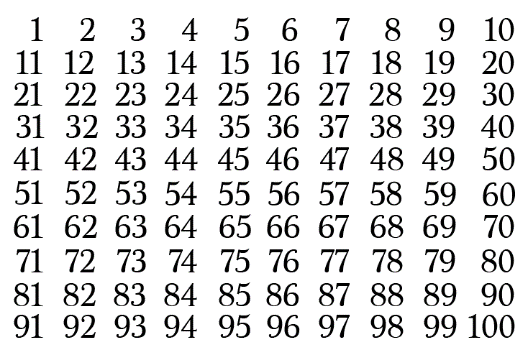
Solution:

; ;

1. Find all primes less than a speciﬁed positive integer n. (let’s say n =100)

Solution: We can find using Sieve of Eratosthenes method.

Step 1: Begin by listing out the numbers from 1 to 100.

1. 

Step 2: Erase all of the multiples of 2, except two itself.

Step 3: Erase all of the multiples of 3, except three itself

Step 4: Erase all of the multiples of 2, 3, 5, and 7, except (2, 3, 5, 7) itself.

Step 5: There will be no multiple of 11, 13, 17, 19, …

Step 6: Erase 1 because it is also not prime number.

Final Step: Count the remaining numbers, which will be 25 as there are 25 prime numbers less than (n = 100)

***Pigeonhole Principle***

**Problem 6**

1. How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of four aces (A) are chosen?

Solution: A standard deck of 52 cards contains 4 aces (A) and 48 other cards. In worst case scenario, we first pick the 48 other cards (non-Ace cards) before picking the 2 aces. This means we need to pick 50 cards to guarantee.

1. How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of each of two different kinds?

Solution: In worst case scenario, we first select four cards of the same kind of each of the four suits. The next card that we select then needs to be different kind and then we require at most 5 cards to gurantee the required condition.

1. How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two cards of the same kind?

Solution: There are 13 different cards among a standard deck of 52 cards. Suppose, we have 13 boxes for each kind. According to the pigeonhole principle, we need 14 cards to guarantee at least two cards of the same kind.

1. How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces (A) and two of the 13 kinds are chosen?

*(Hint: Two cards are of the same kind, if they have same number or figure. They may belong to different suits, e.g. a 3 of club and a 3 of hearts are of the same kind.)*

Solution: In worst case scenario, we first pick the 48 other cards before picking the 2 aces. Thus, we need to pick at least 50 cards to guarantee the required condition.

**Problem 7**

A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red, and 4 pairs of green socks.

1. How many single socks do we have to take out to make sure that we take out two socks with the same color?

Solution: According to the pigeonhole principle, 5 socks can ensure that we take out two socks of the same colors

1. How many single socks do we have to take out to make sure that we take out two socks with different colors?

Solution: The worst case scenario is that we keep grabbing black sock, in which case we can take 12 socks and get only one color. Thus, 13 socks will guarantee at least two colors.

**Problem 8**

We select 38 even positive integers, all less than 1000. Prove that there will be two of them whose difference is at most 26.

Solution: Since we pick even positive integers, the smallest number is 2. Thus we divide (2 – 1000) into ranges: and so on. There are ranges. However, there are 38 selected integers, thus two of them must fall into the same range and their difference must be at most 26.

***Generalized Counting***

**Problem 9**

1. How many possible passwords can be formed from the letters: ***ASSISTANT\_TITAN* ?**
2. Solution:

Thus

1. How many different strings can be made from the letters in ***MISSISSIPPI*,** using all the letters?

Solution:

1. How many different strings can be made from the letters in ***ABRACADABRA***, using all the letters?

Solution:

**Problem 10**

1. How many non-negative integer solutions are there of the equation: **x + y + z + w = 15**?

Solution: As we have four variables and the total required is 15 and there is no condition but non-negative, therefore, we can state that and .

Using the formula

1. How many solutions are there to the equation:

such that .

Solution:

We need to find a new value of r because of the conditions, i.e.

Therefore, we need to find

Now, we have five variables and total must be equal to four. Therefore using the formula

1. How many non-negative integer solutions are there of the equation: **x + y + z + w = 15,** such tha**t** ?

As we know exact value of w therefore, we can subtract its value from the given equation, i.e. And the modified equation will be . As we have some information about z as well, therefore the value of r will get re-modified as and the equation will become . Now we can find the non-negative solutions as and . Therefore,

**Problem 11**

In how many ways can you distribute n pennies to k children if each child is supposed to get at least 5 pennies?

Solution: Since every child is supposed to get at least 5 pennies, therefore, .

Step 1: First child gets 5 coins.

Step 2 : Now, we have coins (because every child must be given 5 coins at least) and we have children.

Step 3: Therefore,

**Problem 12**

A toy shop has 15 airplanes, 15 buses, 17 trains, and 20 bikes in the stock.

1. How many ways are there for a person to take 15 toys home if all the airplanes are identical, all the buses are identical, all the trains are identical and all the bikes are identical?

Solution:

1. How many ways are there for a person to take 15 toys home if all the airplanes are distinct, all the buses are distinct, all the trains are distinct and all the bikes are distinct?

Solution: Total objects (15 + 15 + 17 + 20 = 67) Total ways =

1. How many ways are there for a person to take 25 toys home if all the airplanes are identical, all the buses are identical, all the trains are identical and all the bikes are identical?

Solution:

We have to choose 25 toys, while the equation becomes:

Airplanes + buses + trains + bikes = 25, with airplanes 15, buses , trains , bikes .

Let us represent all types of toys with variables for simplicity.

We can solve it using principle of inclusion-exclusion as done in the class.

Total solutions without any conditions are:

Now the solutions which are not possible (taking compliment of all conditions one-by-one):

Keeping at least aside,

solutions.

Keeping at least aside,

solutions.

Keeping at least aside,

solutions.

Keeping at least aside,

solutions.

Now we need to take ,these have been double counted so we need to subtract these.

This means, we should keep 16+16 = 32 toys aside, before picking up other combination of toys. But it is not possible, as we have to pick maximum 25 toys. So no such solutions possible.

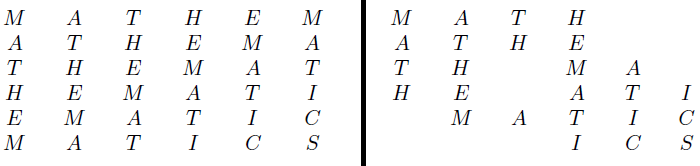
So same argument applies to all other pair-wise intersections as well. Similarly for all triple-intersections and quad-intersections.

**So finally**, net solutions are:

Total solution without conditions – not allowed solutions = 3276 – 220 – 220 – 120 – 35 = 2681 possible solutions.

**Problem 13**

In how many ways can you read off the word MATHEMATICS from the following tables:



Solution:

(Part: left-column) - There are 10 letters in mathematics (as we are standing at M), and each move is either right or down. There must be 5 rights and 5 downs, so there are ways to choose the 5 rights among the 10 moves.

(Part: right-Column) - There are 10 letters in mathematics (as we are standing at M), and each move is either right or down. There must be 5 rights (maximum) and 5 downs, so there are ways to choose the 5 rights among the 10 moves.

***Combinatorial Identities***

**Problem 14**

1. What is the coefficient of **x101y99** in the expansion of **(2x − 3y)200.**

**Solution:**

1. What is the coefficient of x9 in (2 − x)19

**Solution:**

1. Give a formula for the coefficient of xk in the expansion of (10x + 20/x)100, where k is **zero** i.e. it is just a constant. ***(We shall get a term independent of )***.

**Solution:**